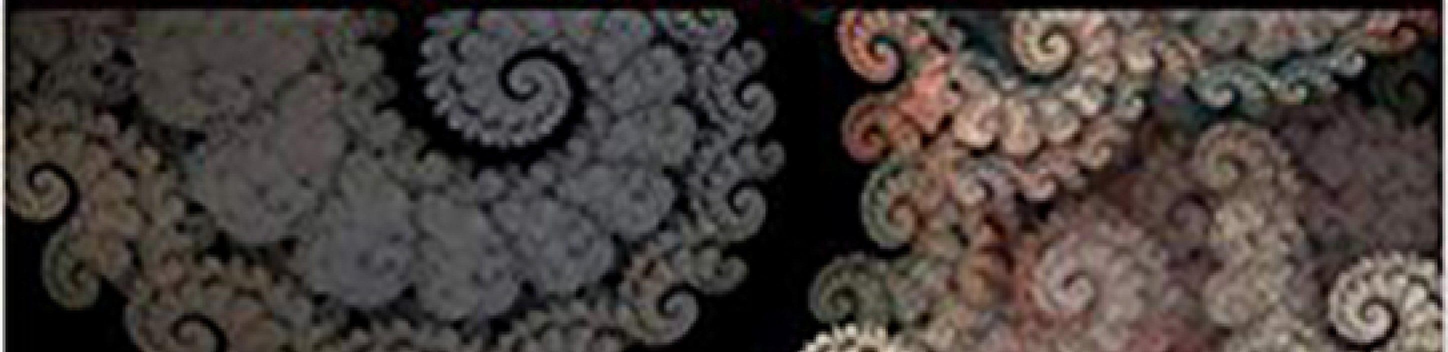
ECONOMIC DYNAMICS IN DISCRETE TIME



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Economic Dynamics: Discrete Time

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Preface

This book is about the analytical and numerical tools for solving dynamic economic problems. The main theme is to introduce recursive methods, which should be in every economist's toolkit. The main idea of recursive methods is to characterize economic dynamics by a set of state variables and a pair of functions. One function, called the state transition function, maps the state and the control (or action) of the model today into the state tomorrow. The other function, called the policy function, maps the state into the control of the model. Economic data may come from either a dynamic optimization problem or a market equilibrium. They can be extremely complicated and hard to analyze. Using a finite number of state variables and a pair of functions to summarize economic data simplifies the analysis significantly.

The ultimate goal of this book is to introduce the reader how to apply recursive methods to a variety of dynamic economic problems. To achieve this goal, I first introduce the theory and numerical methods of solving linear and nonlinear systems of deterministic and stochastic difference equations. These systems can be derived from dynamic optimization or equilibrium conditions. I then introduce the theory and numerical methods of solving dynamic optimization problems. One powerful tool is dynamic programming. Another powerful tool is the maximum principle or the Lagrange method. Though this book focuses on the former tool, the connection between these two tools is discussed, and the latter tool is used whenever it is more convenient.

An important feature of this book is that it combines theoretical foundations with numerical methods. For each topic, I begin with theoretical foundations with explicit definitions and rigorous proofs. I then introduce numerical methods and computer codes to implement these methods. In earlier years, it was quite cumbersome to numerically solve dynamic stochastic general equilibrium (DSGE) models. Students and researchers found it hard to replicate numerical results in published papers. This has been changed since the 1990s. Researchers have developed efficient numerical methods to solve medium to large scale DSGE models and to perform Bayesian estimation of these models. These methods have been made popular since the launch of Dynare in the late 1990s. Dynare is a software platform for handling a wide class of economic models, in particular, DSGE models and overlapping generations (OLG) models. A large part of the book is to introduce the reader how to use Dynare to numerically solve DSGE models and to perform Bayesian estimation of DSGE models.

The book consists of five parts. Part I presents the theory of dynamical systems and numerical methods of solving dynamical systems. This part lays out the foundation for other parts of the book. Chapters 1 and 2 introduce the analytical and numerical tools of solving deterministic and stochastic linear and nonlinear systems of difference equations. These two chapters also introduce how to use Dynare to implement the numerical methods. Chapter 3 introduces the theory of Markov processes and their convergence. This theory is important for setting up dynamic optimization problems. Chapter 4 presents ergodic theory and stationary processes. Ergodic theory is important for understanding long-run properties of stochastic processes and has many applications in econometrics.

Part II introduces the theory and applications of dynamic optimization. Chapter 5 introduces how to set up a dynamic optimization problem in terms of the Markov decision process model. Chapters 6 and 7 present the theory of finite- and infinite-horizon dynamic programming, respectively. These two chapters analyze the Bellman equation and properties of the value function and of the policy function. The maximum principle and its relation to dynamic programming are also discussed. Chapter 8 provides a variety of applications of dynamic programming, including discrete choice, consumption/saving, portfolio choice, inventory, and investment. Chapter 9 introduces linear-quadratic models and robust control. Applications to policy analysis are discussed. In addition, the notion of commitment and time inconsistency is presented. Chapter 10 presents filtering and control under partial information. In particular, this chapter introduces the Kalman filter, which is important for Bayesian estimation studied in Chapter 15. Chapter 11 presents numerical methods for solving dynamic programming problems. Projection methods, perturbation methods, and value function iteration methods are stressed. Chapter 12 introduces methods of structural estimation of dynamic programming problems. It focuses on the generalized method of moments, the maximum likelihood method, and the simulated method of moments.

Part III presents equilibrium analysis of a variety of core models in macroeconomics. For each model, I start by describing its basic structure and then discuss its various extensions. Chapter 13 describes models of complete markets pure exchange economies. These models are useful for understanding consumption insurance and asset pricing. Chapter 14 introduces neoclassical growth models. These models are the cornerstone of modern macroeconomics. Chapter 15 introduces how to use Dynare to implement Bayesian estimation of DSGE models. Chapter 16 presents overlapping generations models. These models are fundamental in public finance and can also generate asset price bubbles. Chapter 17 studies a particular type of incomplete markets models, the Bewley-Aiyagari-Huggett model. In this model, market incompleteness comes from missing markets. Chapter 18 introduces search and matching models. These models are useful for understanding unemployment. Chapter 19 presents the dynamic New Keynesian models. These models are useful for understanding inflation and monetary policy.

Part IV studies three additional topics. Chapter 20 describes recursive utility. Recursive utility has become increasingly popular in finance and macroeconomics because recursive methods, such as dynamic programming, can be tractably applied. I embed a variety of static utility models from decision theory in the dynamic framework of recursive utility. These static models typically depart from the rational expectations hypothesis and are motivated by experimental evidence such as the Allais paradox and the Ellsberg paradox. Embedding them in the framework of recursive utility allows them to be used to address many dynamic asset pricing puzzles.

Chapter 21 presents dynamic games and credible government policies. The main tool of the analysis is developed by Abreu, Pearce, and Stacchetti (henceforth, APS) (1990). This tool is a significant breakthrough in the application of recursive methods. Unlike the traditional method of dynamic programming based on the Bellman equation, the object of the APS method is sets, instead of functions. The key idea is to use the continuation value as a state variable to make the problem recursive. Chapter 22 introduces recursive contracts. Models with incentive problems are hard to analyze because of the history dependence of contracts. Spear and Srivastava (1987), Thomas and Worrall (1988), and APS (1990) make a significant breakthrough by incorporating the continuation value promised by the principal to the agent as a state variable in order to make the problem recursive.

Part V contains four mathematical appendixes which present basic concepts and results from linear algebra, real and functional analysis, convex analysis, and measure and probability theory. I have tried to make this book self-contained by collecting all necessary mathematical concepts and results beyond the undergraduate analysis, linear algebra, and probability theory in the appendixes.

This book uses many Matlab programs to solve various examples and exercises. These programs are referred to in a special index at the end of the book. They can be downloaded from the website ???...

Other books whose treatments overlap with some of the topics covered here include Sargent (1987), Blanchard and Fischer (1989), Stokey, Lucas, and Prescott (1989), Cooley (1995), Farmer (1993), Azariadis (1993), Chow (1997), Judd (1998), Miranda and Fackler (2002), Adda and Cooper (2003), Woodford (2003), Hansen and Sargent (2008), Acemoglu (2008), Walsh (2010), DeJong and Dave (2011), Romer (2012), and Ljungqvist and Sargent (2012).

Each of the above books has its own aims and themes. What is new about this book is the emphasis on the balance between analytical and numerical methods and the up-to-date treatment of the recent developments in economic dynamics. Theoretical results are stated as propositions or theorems and proved rigorously. Numerical methods are presented with theoretical foundations and their computer implementations are provided whenever possible. Since the late 1990s, the field of economic dynamics has developed rapidly. I have tried to incorporate some recent developments, such as numerical methods for solving linear and nonlinear rational expectations models, robust control, Bayesian estimation of DSGE models, perturbation methods, projection methods, asset price bubbles, recursive models of ambiguity and robustness, recursive utility, and recursive contracts.

This book focuses on the analytical and numerical tools, rather than empirical applications. Thus, I do not present data analysis and do not discuss how to tie the theory to the data. The book focuses exclusively on discrete-time models. Many basic ideas for discrete-time models can be applied to continuous time. I decide to treat continuous-time problems elsewhere, although continuous-time models typically admit closed-form solutions and are analytically convenient in many contexts, especially, in the theory of finance and economic growth. I also leave out some important topics such as endogenous growth, fiscal policy, and optimal taxation.

While most applications in the book focus on macroeconomics, the theory and methods should be valuable in other fields of economics. For example, the theory and numerical methods of dynamic programming can be applied to analyze any dynamic optimization problems in any field of economics. The treatment of dynamic games and recursive contracts in Chapters 21 and 22 should be of interest to game theorists. The introduction of recursive utility in Chapter 20 should be valuable in decision theory. The discussion of asset pricing in Chapters 13 and 20 is useful in finance.

This book can be used for various courses. Here are some examples:

- A one-semester first-year graduate macroeconomics course: Chapters 1-3, 5-7, and 13-16.

- A second-semester first-year graduate macroeconomics course: Chapters 8-9, 11, 17-19, and any one from Chapters 20-22.

- A graduate course on economic dynamics: The core materials are in Parts I and II. Instructors can select any chapters from the remaining parts depending on the students' interest.

- A second-year graduate course on topics in macroeconomics or financial economics: Any chapters from Parts III and IV.

I have benefited from research collaboration over the years with many coauthors, including Rui Albuquerque, Dan Bernhardt, Hui Chen, Larry Epstein, Zhigang Feng, Francois Gourio, Xin Guo, Dirk Hackbarth, Nengjiu Ju, Larry Kotlikoff, Erwan Morellec, Adrian Peralta-Alva, Manuel Santos, Hayashi Takashi, Neng Wang, Pengfei Wang, Danyang Xie, Lifang Xu, Zhiwei Xu, and Hao Zhou.

This book is based on my lecture notes for the graduate course, Economic Dynamics, I have taught at Boston University for about 9 years. I thank Bob King for suggesting and encouraging me to create this course. I also thank many students at Boston University and Central University of Finance and Economics for comments on the book. I would especially like to thank Brittany Baumann, Chenyu Hui, Yue Jiang, Hyosung Kwon, Xiao Yang, and Fan Zhuo. I appreciate the comments of outside reviewers and the editorial staff at the MIT Press. Finally, I deeply appreciate the support from my wife, Qian Jiang, during my writing of this book. Without her support, the book cannot be completed.

Part I

Dynamical Systems

The dynamics of economic variables are typically described by the following system of p-order difference equations:

$$x_{t+p} = f\left(x_t, x_{t+1}, ..., x_{t+p-1}, z_t, z_{t+1}, ..., z_{t+p-1}\right), \tag{1}$$

where $f : \mathbb{R}^{np} \times \mathbb{R}^{n_z p} \rightarrow \mathbb{R}^n$, $x_t \in \mathbb{R}^n$, $z_t \in \mathbb{R}^{n_z}$ for all $t = 0, 1, ...,$ and n, n_z and p are natural numbers. The vector z_t consists of exogenously given forcing variables. We need to impose certain initial or terminal conditions to solve (1). These conditions typically depend on the economic problems at hand. By an appropriate change of variables, we can often transform (1) into a system of first-order difference equations. If the sequence $\{z_t\}$ is deterministic, then (1) is a deterministic system. In Chapter 1, we study this case. If $\{z_t\}$ is a stochastic process, then (1) is a stochastic system. In this case, we introduce an information structure and require f to satisfy certain measurability condition. In a dynamic economy, economic agents must form expectations about future variables. If system (1) characterizes a rational expectations equilibrium, we must introduce conditional expectations into this system. We study the stochastic case in Chapter 2. Researchers typically use a recursive approach to study dynamic equilibria. Under this approach, equilibrium variables typically satisfy certain Markov properties. In Chapter 3, we study Markov processes and their convergence. A central issue is the existence and uniqueness of a stationary distribution. In Chapter 4, we discuss ergodic theory and its applications to stationary processes. We establish several strong laws of large numbers for stationary processes and for Markov processes in particular.

Chapter 1

Deterministic Difference Equations

In this chapter, we focus on deterministic dynamics characterized by systems of first-order linear difference equations. We distinguish between singular and nonsingular systems because different solution methods are applied to these two cases. We also introduce lag operators and apply them to solve second-order linear difference equations. Finally, we provide a brief introduction to nonlinear dynamics.

1.1 Scalar First-Order Linear Equations

Consider the following scalar first-order linear difference equation:

$$x_{t+1} = bx_t + cz_t, \ t \geq 0, \tag{1.1}$$

where x_t, b, c, and z_t are all real numbers. Assume that $\{z_t\}$ is an exogenously given bounded sequence. If z_t is constant for each t, then (1.1) is **autonomous**. When $cz_t = 0$ for all t, we call (1.1) a **homogeneous** difference equation. These concepts can be generalized to systems of high-order difference equations introduced later.

In the autonomous case, we may suppose $z_t = 1$ for all t in (1.1) without loss of generality. We then obtain

$$x_{t+1} = bx_t + c. \tag{1.2}$$

A particular solution to this difference equation is a constant solution $x_t = \bar{x}$ for all t, where

$$\bar{x} = \frac{c}{1-b}, \text{ for } b \neq 1.$$

This solution is called a **stationary point** or **steady state**. One can verify that the general solution to (1.2) is given by

$$x_t = (x_0 - \bar{x})\, b^t + \bar{x}. \tag{1.3}$$

We are interested in the long-run behavior of this solution:

- If $|b| < 1$, then the solution in (1.3) converges asymptotically to the steady state $\bar{x}$ for any initial value x_0. In this case, we call $\bar{x}$ a **globally asymptotically stable** steady state. If x_0 is not exogenously given, then the solution is **indeterminate**. Starting from any initial value x_0, equation (1.3) gives a solution to (1.2).

- If $|b| > 1$, then the solution in (1.3) explodes or is unstable for any given initial value $x_0 \neq \bar{x}$, unless $x_0 = \bar{x}$. In this case, we often assume that x_0 is unknown and solve for the entire path of x_t. The only stable solution is $x_t = \bar{x}$ for all $t \geq 0$.

In the nonautonomous case, we may solve for $\{x_t\}$ in two ways depending on whether or not the initial value x_0 is exogenously given. First, consider the case in which x_0 is exogenously given. Then we solve for x_t backward by repeated substitution to obtain the backward-looking solution:

$$x_t = c\sum_{j=0}^{t-1} b^j z_{t-1-j} + b^t x_0. \tag{1.4}$$

If $|b| < 1$, then

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} c \sum_{j=0}^{t-1} b^j z_{t-1-j}, \tag{1.5}$$

where a finite limit exists because we assume $\{z_t\}$ is a bounded sequence. Thus, for any given initial value x_0, the difference equation in (1.1) has a solution for $\{x_t\}$, which converges to a finite limit in (1.5). We call this limit a **generalized steady state**. It is globally asymptotically stable. If $|b| > 1$, then (1.4) shows that $\{x_t\}$ diverges. If $|b| = 1$, then the solution does not converge to a finite limit unless $\sum_{j=0}^{\infty} z_j$ is finite. Even if a finite limit exists, it depends on the initial condition x_0 so that the solution is not globally stable.

Second, suppose that x_0 is not exogenously given. For example, x_t represents an asset's price. Let b be the gross return and $-cz_t > 0$ be the asset's dividends. Then equation (1.1) is an asset pricing equation. We may solve for x_t forward by repeated substitution:

$$x_t = \left(\frac{1}{b}\right)^T x_{t+T} - \frac{c}{b} \sum_{j=0}^{T-1} \left(\frac{1}{b}\right)^j z_{t+j}, \tag{1.6}$$

for any $T \geq 1$. Taking $T \to \infty$ and assuming the **transversality condition (**or **no-bubble condition)**,

$$\lim_{T\to\infty} \left(\frac{1}{b}\right)^T x_{t+T} = 0, \tag{1.7}$$

we obtain the forward-looking solution:

$$x_t = -\frac{c}{b} \sum_{j=0}^{\infty} \left(\frac{1}{b}\right)^j z_{t+j}. \tag{1.8}$$

If $|b| > 1$, then the above infinite sum is finite since $\{z_t\}$ is a bounded sequence. Clearly, the above solution also satisfies the transversality condition (1.7). This solution is **stable** in the sense that x_t is bounded for all $t \geq 0$.

If we remove the transversality condition (1.7), then (1.1) admits many unstable solutions. Let x_t^* denote the solution given by (1.8). Then for any B_t satisfying

$$B_{t+1} = bB_t, \tag{1.9}$$

the expression, $x_t = x_t^* + B_t$, is a solution to (1.1). We often call x_t^* the fundamental solution and B_t a bubble. The bubble grows at the gross rate b.

If $|b| < 1$, then the infinite sum in (1.8) is unlikely to converge in general. There is an infinity of bubble solutions, which are globally stable rather than exploding. For example, let $z_t = 1$ for all t, then the expression below is a solution to (1.1):

$$x_t = \frac{c}{1-b} + B_t,$$

where B_t satisfies (1.9) and B_0 is any given value. This is related to indeterminacy discussed earlier for the autonomous system. Theorem 1.4.3 studied later will consider more general cases.

Example 1.1.1 *Asset prices under adapted versus rational expectations.*

Consider the following asset pricing equation:

$$p_t = \frac{{}_t p_{t+1}^e + d}{R}, \tag{1.10}$$

where d represents constant dividends, R is the gross return on the asset, p_t is the asset price in period t, and ${}_t p_{t+1}^e$ is investors' period-t forecast of the price in period $t+1$. An important question is how to form this forecast. According to the adapted expectations hypothesis, the forecast satisfies

$${}_t p_{t+1}^e = (1-\lambda)\ {}_{t-1} p_t^e + \lambda p_t, \tag{1.11}$$

where $\lambda \in (0,1)$. This means that investors' current forecast of the next-period price is equal to a weighted average of the current price and the *previous-period forecast of the current price. Using equation (1.10) to substitute for* ${}_tp^e_{t+1}$ *and* ${}_{t-1}p^e_t$ *into equation (1.11), we obtain*

$$Rp_t - d = (1 - \lambda)\,(Rp_{t-1} - d) + \lambda p_t.$$

Simplifying yields:

$$(R - \lambda)\,p_t = (1 - \lambda)\,Rp_{t-1} + \lambda d.$$

Solving this equation backward until time 0*, we obtain the backward-looking solution:*

$$p_t = a^t p_0 + \frac{\left(1 - a^t\right)d}{R - 1},$$

where $a = \frac{R(1-\lambda)}{R-\lambda}$*. We need to assign an exogenously given initial value* p_0*. For this solution to be stable, we must assume that* $|a| < 1$*. In this case,* p_t *converges to its steady state value* $\bar{p} = d/\,(R-1)\,,$ *starting at any initial value* p_0*.*

We next turn to the case under rational expectations. In a deterministic model, rational expectations mean perfect foresight in the sense that ${}_tp^e_{t+1} = p_{t+1}$*. That is, investors' rational forecast of the future price is identical to its true value. In this case, we rewrite (1.10) as:*

$$p_t = \frac{p_{t+1} + d}{R}, \tag{1.12}$$

Solving this equation forward, we obtain:

$$p_t = \frac{d}{R - 1} + \lim_{T\to\infty} \frac{p_{t+T}}{R^T}.$$

Ruling out bubbles, we obtain the forward-looking solution, $p_t = d/\,(R-1)\,,$ $t \geq 0$*. This means that the stock price in each period is always equal to the constant fundamental value.*

Example 1.1.2 *Dividend taxes*

Suppose that dividends are taxed at the constant rate τ_1 *from time* 0 *to time* T. *From time* $T+1$ *on, the dividend tax rate is increased to* τ_2 *forever. Suppose that this policy is publicly announced at time* 0. *What will happen to the stock price at time* 0? *Given the rational expectations hypothesis, we solve the price at time* T :

$$p_T = \frac{(1-\tau_2)\, d}{R-1}.$$

At time 0, *we use equation (1.6) to derive the forward-looking solution:*

$$\begin{aligned} p_0 &= \frac{1}{R^T} p_T + \frac{1}{R} \sum_{j=0}^{T-1} \frac{(1-\tau_1)\, d}{R^j} \\ &= \frac{(1-\tau_1)\, d}{R-1} + \frac{1}{R^T} \left(\frac{(1-\tau_2)\, d}{R-1} - \frac{(1-\tau_1)\, d}{R-1} \right). \end{aligned}$$

Thus, the stock price drops immediately at time 0 *and then continuously declines until it reaches the new fundamental value. Figure 1.1 shows a numerical example. The dashed and solid lines represent the price path without and with tax changes, respectively.*

In this section, we have shown that two conditions are important for solving a linear difference equation: (i) whether the initial value is given; and (ii) whether the coefficient b is smaller than one in absolute value. We will show below that similar conditions apply to general multivariate linear systems. In particular, the first condition determines whether the variable x_t is predetermined, and the second condition corresponds to whether the eigenvalue is stable.

1.2 Lag Operators

Lag operators provide a powerful tool for solving difference equations. They are also useful for analyzing economic dynamics and time series econometrics. We now introduce these operators.[1]

[1] We refer readers to Sargent (1987) and Hamilton (1994) for a further introduction.

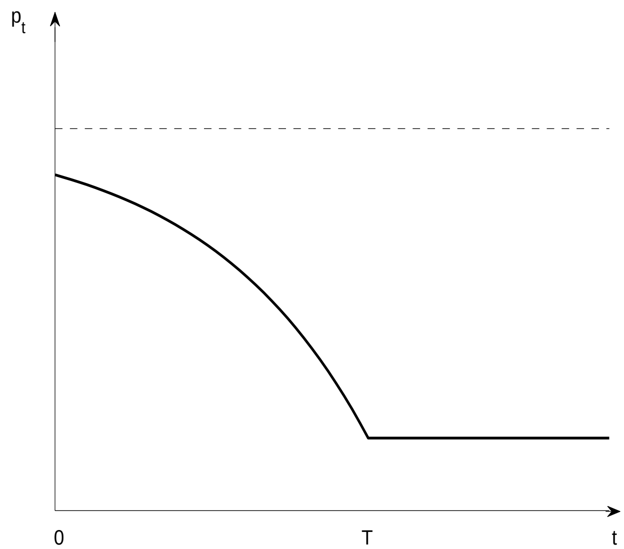

Figure 1.1: **The impact of dividend taxes on the stock price.**

Consider a sequence $\{X_t\}_{t=-\infty}^{\infty}$. The lag operator $\mathbf{L}$ on this sequence is defined by

$$\mathbf{L}X_t = X_{t-1},\ \mathbf{L}^n X_t = X_{t-n},\ \text{for all } n = ..., -2, -1, 0, 1, 2, ...$$

In addition, $\mathbf{L}^n c = c$ for any constant c that is independent of time. The following formulas are useful in applications:

$$\frac{1}{1-\lambda \mathbf{L}^n} = \sum_{j=0}^{\infty} \lambda^j \mathbf{L}^{nj},$$

$$\frac{1}{(1-\lambda \mathbf{L}^n)^2} = \sum_{j=0}^{\infty} (j+1)\,\lambda^j \mathbf{L}^{nj},$$

for $|\lambda| < 1$, and

$$(I - A\mathbf{L}^n)^{-1} = \sum_{j=0}^{\infty} A^j \mathbf{L}^{nj},$$